2.4 Complex Numbers

The Imaginary Unit $i$

Some quadratic equations have no real solutions. For instance, the quadratic equation $x^2 + 1 = 0$ has no real solution because there is no real number $x$ that can be squared to produce $-1$. To overcome this deficiency, mathematicians created an expanded system of numbers using the imaginary unit $i$, defined as

$$i = \sqrt{-1}$$

where $i^2 = -1$. By adding real numbers to real multiples of this imaginary unit, you obtain the set of complex numbers. Each complex number can be written in the standard form $a + bi$. For instance, the standard form of the complex number $\sqrt{-9} - 5$ is $-5 + 3i$ because

$$\sqrt{-9} - 5 = \sqrt{3^2(-1)} - 5 = 3\sqrt{-1} - 5 = -5 + 3i.$$ 

In the standard form $a + bi$, the real number $a$ is called the real part of the complex number and the number $bi$ (where $b$ is a real number) is called the imaginary part of the complex number.

Definition of a Complex Number

If $a$ and $b$ are real numbers, the number $a + bi$ is a complex number, and it is said to be written in standard form. If $b = 0$, the number $a + bi = a$ is a real number. If $b \neq 0$, the number $a + bi$ is called an imaginary number. A number of the form $bi$, where $b \neq 0$, is called a pure imaginary number.

The set of real numbers is a subset of the set of complex numbers, as shown in Figure 2.27. This is true because every real number $a$ can be written as a complex number using $b = 0$. That is, for every real number $a$, you can write $a = a + 0i$.

Equality of Complex Numbers

Two complex numbers $a + bi$ and $c + di$, written in standard form, are equal to each other if and only if $a = c$ and $b = d$.
Operations with Complex Numbers

To add (or subtract) two complex numbers, you add (or subtract) the real and imaginary parts of the numbers separately.

**Addition and Subtraction of Complex Numbers**

If \(a + bi\) and \(c + di\) are two complex numbers written in standard form, their sum and difference are defined as follows.

**Sum:** \((a + bi) + (c + di) = (a + c) + (b + d)i\)

**Difference:** \((a + bi) - (c + di) = (a - c) + (b - d)i\)

The **additive identity** in the complex number system is zero (the same as in the real number system). Furthermore, the **additive inverse** of the complex number \(a + bi\) is

\[-(a + bi) = -a - bi.\]

So, you have

\[(a + bi) + (-a - bi) = 0 + 0i = 0.\]

**EXAMPLE 1** Adding and Subtracting Complex Numbers

a. \((3 - i) + (2 + 3i) = 3 - i + 2 + 3i\)
   \[= 3 + 2 - i + 3i\]
   \[= (3 + 2) + (-1 + 3)i\]
   \[= 5 + 2i\]

b. \(2i + (-4 - 2i) = 2i - 4 - 2i\)
   \[= -4 + 2i - 2i\]
   \[= -4\]

c. \(3 - (-2 + 3i) + (-5 + i) = 3 + 2 - 3i - 5 + i\)
   \[= 3 + 2 - 5 - 3i + i\]
   \[= 0 - 2i\]
   \[= -2i\]

d. \((3 + 2i) + (4 - i) - (7 + i) = 3 + 2i + 4 - i - 7 - i\)
   \[= 3 + 4 - 7 + 2i - i - i\]
   \[= 0 + 0i\]
   \[= 0\]

In Examples 1(b) and 1(d) note that the sum of complex numbers can be a real number.
Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

**Exploration**
Complete the table:

<table>
<thead>
<tr>
<th>$i^1$</th>
<th>$i^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$i^3$</td>
<td>$-i$</td>
</tr>
<tr>
<td>$i^4$</td>
<td>$1$</td>
</tr>
<tr>
<td>$i^5$</td>
<td>$i$</td>
</tr>
<tr>
<td>$i^6$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$i^7$</td>
<td>$i$</td>
</tr>
<tr>
<td>$i^8$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$i^9$</td>
<td>$i$</td>
</tr>
<tr>
<td>$i^{10}$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

What pattern do you see? Write a brief description of how you would find $i$ raised to any positive integer power.

**EXAMPLE 2** Multiplying Complex Numbers

a. $(i)(-3i) = -3i^2$
   
   $= -3(-1)$
   
   $= 3$

b. $\sqrt{-4} \cdot \sqrt{-16} = (2i)(4i)$
   
   $= 8i^2$
   
   $= 8(-1)$
   
   $= -8$

c. $(2 - i)(4 + 3i) = 8 + 6i - 4i - 3i^2$
   
   $= 8 + 6i - 4i - 3(-1)$
   
   $= 8 + 3 + 6i - 4i$
   
   $= 11 + 2i$

d. $(3 + 2i)(3 - 2i) = 9 - 6i + 6i - 4i^2$
   
   $= 9 - 4(-1)$
   
   $= 9 + 4$
   
   $= 13$

e. $(3 + 2i)^2 = 9 + 6i + 6i + 4i^2$
   
   $= 9 + 4(-1) + 12i$
   
   $= 9 - 4 + 12i$
   
   $= 5 + 12i$

**STUDY TIP**

Note in Example 2(b) that

$\sqrt{-4} \cdot \sqrt{-16} \neq \sqrt{(-4)(-16)}$

$= \sqrt{64}$

$= 8$. 
Complex Conjugates and Division

Notice in Example 2(d) that the product of two complex numbers can be a real number. This occurs with pairs of complex numbers of the form \(a + bi\) and \(a - bi\), called complex conjugates.

\[
(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 \\
= a^2 - b^2(-1) \\
= a^2 + b^2
\]

To find the quotient of \(a + bi\) and \(c + di\) where \(c\) and \(d\) are not both zero, multiply the numerator and denominator by the conjugate of the denominator to obtain

\[
\frac{a + bi}{c + di} = \frac{a + bi(c - di)}{c + di(c - di)} \\
= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}.
\]

**EXAMPLE 3** Dividing Complex Numbers

\[
\frac{1}{1 + i} = \frac{1}{1 + i} \left(\frac{1 - i}{1 - i}\right) \quad \text{Multiply numerator and denominator by conjugate of denominator.}
\]

\[
= \frac{1 - i}{1^2 - i^2} \quad \text{Expand.}
\]

\[
= \frac{1 - i}{1 - (-1)} \\
= \frac{1 - i}{2} \quad \text{i}^2 = -1
\]

\[
= \frac{1}{2} - \frac{1}{2}i \quad \text{Simplify.}
\]

\[
= \frac{1}{2} - \frac{1}{2}i \quad \text{Write in standard form.}
\]

**EXAMPLE 4** Dividing Complex Numbers

\[
\frac{2 + 3i}{4 - 2i} = \frac{2 + 3i(4 + 2i)}{4 - 2i(4 + 2i)} \quad \text{Multiply numerator and denominator by conjugate of denominator.}
\]

\[
= \frac{8 + 4i + 12i + 6i^2}{16 - 4i^2} \quad \text{Expand.}
\]

\[
= \frac{8 + 6 + 16i}{16 + 4} \quad i^2 = -1
\]

\[
= \frac{16 + 16i}{20} \quad \text{i}^3 = -1
\]

\[
= \frac{1}{2} + \frac{4}{5}i \quad \text{Simplify.}
\]

\[
= \frac{1}{2} + \frac{4}{5}i \quad \text{Write in standard form.}
\]
Applications

Most applications involving complex numbers are either theoretical or very technical, and are therefore not appropriate for inclusion in this text. However, to give you some idea of how complex numbers can be used in applications, we give a general description of their use in fractal geometry.

To begin, consider a coordinate system called the complex plane. Just as every real number corresponds to a point on the real number line, every complex number corresponds to a point in the complex plane, as shown in Figure 2.28. In this figure, note that the vertical axis is the imaginary axis and the horizontal axis is the real axis. The point that corresponds to the complex number \( a + bi \) is \((a, b)\).

**EXAMPLE 5  Plotting Complex Numbers**

Plot each complex number in the complex plane.

<table>
<thead>
<tr>
<th>a. ( 2 + 3i )</th>
<th>b. (-1 + 2i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>c. ( 4 )</td>
<td>d. (-3i)</td>
</tr>
</tbody>
</table>

**Solution**

a. To plot the complex number \( 2 + 3i \), move (from the origin) two units to the right on the real axis and then three units up, as shown in Figure 2.29. In other words, plotting the complex number \( 2 + 3i \) in the complex plane is comparable to plotting the point \((2, 3)\) in the Cartesian plane.

b. The complex number \(-1 + 2i\) corresponds to the point \((-1, 2)\), as shown in Figure 2.29.

c. The complex number \(4\) corresponds to the point \((4, 0)\), as shown in Figure 2.29.

d. The complex number \(-3i\) corresponds to the point \((0, 3)\), as shown in Figure 2.29.

In the hands of a person who understands fractal geometry, the complex plane can become an easel on which stunning pictures, called fractals, can be drawn. The most famous such picture is called the Mandelbrot Set, named after the Polish-born mathematician Benoit Mandelbrot. To draw the Mandelbrot Set, consider the following sequence of numbers.

\[ c, c^2 + c, (c^2 + c)^2 + c, ((c^2 + c)^2 + c)^2 + c, \ldots \]

The behavior of this sequence depends on the value of the complex number \( c \). For some values of \( c \) this sequence is bounded, which means that all elements in the sequence are less than some fixed number \( N \). For other values it is unbounded, which means that the elements in the sequence become infinitely large. If the sequence is bounded, the complex number \( c \) is in the Mandelbrot Set; if the sequence is unbounded, the complex number \( c \) is not in the Mandelbrot Set.
EXAMPLE 6 Members of the Mandelbrot Set

a. The complex number $-2$ is in the Mandelbrot Set because for $c = -2$, the corresponding Mandelbrot sequence is $-2, 2, 2, 2, 2, 2, \ldots$, which is bounded.

b. The complex number $i$ is also in the Mandelbrot Set because for $c = i$, the corresponding Mandelbrot sequence is

$$i, \ -1 + i, \ -i, \ -1 + i, \ -i, \ -1 + i, \ldots$$

which is bounded.

c. The complex number $1 + i$ is not in the Mandelbrot Set because for $c = 1 + i$, the corresponding Mandelbrot sequence is

$$1 + i, \ 1 + 3i, \ -7 + 7i, \ 1 - 97i, \ -9407 - 193i,$$

$$88454491 + 3631103i, \ldots$$

which is unbounded.

With this definition, a picture of the Mandelbrot Set would have only two colors: one color for points that are in the set (the sequence is bounded), and one for points that are outside the set (the sequence is unbounded). Figure 2.30 shows a black and yellow picture of the Mandelbrot Set. The points that are black are in the Mandelbrot Set and the points that are yellow are not.

![Mandelbrot Set](image)

Figure 2.30

To add more interest to the picture, computer scientists discovered that the points that are not in the Mandelbrot Set can be assigned a variety of colors, depending on "how quickly" their sequences diverge. Figure 2.31 shows three different appendages of the Mandelbrot Set. (The black portions of the picture represent points that are in the Mandelbrot Set.)

Figure 2.32 shows another type of fractal. From this picture, you can see why fractals have fascinated people since their discovery (around 1980). The fractal shown was produced on a graphing calculator.

![Fractal Fern](image)

Figure 2.32 A Fractal Fern
2.4 Exercises

In Exercises 1–4, solve for \(a\) and \(b\).

1. \(a + bi = -9 + 4i\)
2. \(a + bi = 12 + 5i\)
3. \((a - 1) + (b + 3)i = 5 + 8i\)
4. \((a + 6) + 2bi = 6 - 5i\)

In Exercises 5–14, write in standard form.

5. \(4 + \sqrt{-25}\)
6. \(3 + \sqrt{-9}\)
7. 12
8. 42
9. \(-5i + i^2\)
10. \(-3i^2 + i\)
11. \((\sqrt{-75})^2\)
12. \((\sqrt{-4})^2 - 7\)
13. \(\sqrt{-0.09}\)
14. \(\sqrt{-0.0004}\)

In Exercises 15–24, perform the addition or subtraction and write the result in standard form.

15. \((4 + i) + (7 - 2i)\)
16. \((11 - 2i) + (-3 + 6i)\)
17. \((-1 + \sqrt{-8}) + (8 - \sqrt{-50})\)
18. \((7 + \sqrt{-18}) - (3 + 3\sqrt{2}i)\)
19. \(13i - (14 - 7i)\)
20. \(22 + (-5 + 8i) + 10i\)
21. \(-\left(\frac{3}{4} + \frac{3}{4}i\right) + \left(\frac{3}{4} + \frac{11}{3}i\right)\)
22. \(-\left(\frac{3}{4} + \frac{3}{4}i\right) - \left(\frac{3}{6} - \frac{5}{6}i\right)\)
23. \((1.6 + 3.2i) + (-5.8 + 4.3i)\)
24. \(-(-3.7 - 12.8i) - (6.1 - \sqrt{-24.5})\)

In Exercises 25–36, perform the operation and write the result in standard form.

25. \(\sqrt{-6} \cdot \sqrt{-2}\)
26. \(\sqrt{-5} \cdot \sqrt{-10}\)
27. \((\sqrt{-10})^2\)
28. \((\sqrt{-75})^2\)
29. \((1 + i)(3 - 2i)\)
30. \((6 - 2i)(2 - 3i)\)
31. \(6i(5 - 2i)\)
32. \(-8i(9 + 4i)\)
33. \((\sqrt{14} + \sqrt{10})(\sqrt{14} - \sqrt{10})i\)
34. \((3 + \sqrt{-5})(7 - \sqrt{-10})\)
35. \((4 + 5i)^2\)
36. \((1 - 2i)^2 - (1 + 2i)^2\)

37. **Error Analysis** Describe the error.
\[\sqrt{-6}\sqrt{-6} = \sqrt{(-6)(-6)} = \sqrt{36} = 6 \times\]

38. **Error Analysis** Describe the error.
\[-i(\sqrt{-4} - 1) = -i(4i - 1) \times\]
\[= -4i^2 - i \times\]
\[= 4 - i\]

In Exercises 39–46, find the product of the number and its conjugate.

39. \(4 + 3i\)
40. \(8 - 12i\)
41. \(-6 - \sqrt{5}i\)
42. \(-3 + \sqrt{2}i\)
43. \(22i\)
44. \(\sqrt{-13}\)
45. \(3 - \sqrt{-2}\)
46. \(1 + \sqrt{-8}\)

In Exercises 47–58, perform the operation and write the result in standard form.

47. \(\frac{6}{i}\)
48. \(\frac{5}{i}\)
49. \(\frac{4}{4 - 5i}\)
50. \(\frac{3}{1 - i}\)
51. \(\frac{2 + i}{2 - i}\)
52. \(\frac{8 - 7i}{1 - 2i}\)
53. \(\frac{6 - 7i}{i}\)
54. \(\frac{8 + 20i}{2i}\)
55. \(\frac{1}{(4 - 5i)^2}\)
56. \(\frac{1}{(2 - 3i)(5i)}\)
57. \(\frac{2}{1 + i} - \frac{3}{1 - i}\)
58. \(\frac{2i}{2 + i} + \frac{5}{2 - i}\)

In Exercises 59–66, simplify the complex number and write it in standard form.

59. \(-6i^3 + i^2\)
60. \(4i^2 - 2i^3\)
61. \(-5i^5\)
62. \((-i)^3\)
63. \((\sqrt{-75})^3\)
64. \((\sqrt{-2})^6\)
65. \(\frac{1}{i^3}\)
66. \(\frac{1}{(2i)^3}\)

In Exercises 67–70, determine the complex number shown in the complex plane.

67. Imaginary axis
68. Imaginary axis

[Diagram showing a complex plane with points marked at certain coordinates]
In Exercises 71–74, plot the complex number in the complex plane.

71. $4 - 5i$
72. $3i$
73. $-6$
74. $-7 + 2i$

Fractals  In Exercises 75–80, find the first six terms in the following sequence.

c, $e^2 + c$, $(e^2 + c)^2 + c$, $[(e^2 + c)^2 + c]^2 + c$, ...

From these terms, do you think the given complex number is in the Mandelbrot Set? Explain your reasoning.

75. $c = 0$
76. $c = 2$
77. $c = \frac{1}{2}$
78. $c = -i$
79. $c = 1$
80. $c = -1$

81. Cube each complex number. What do you notice?
   (a) 2  (b) $-1 + \sqrt{3}i$  (c) $-1 - \sqrt{3}i$

82. Raise each number to the fourth power.
   (a) 2  (b) $-2$  (c) $2i$  (d) $-2i$

83. Impedance  The opposition to current in an electrical circuit is called its impedance. The impedance in a parallel circuit with two pathways satisfies the equation

$$\frac{1}{Z} = \frac{1}{z_1} + \frac{1}{z_2}$$

when $z_1$ is the impedance (in ohms) of pathway 1 and $z_2$ is the impedance (in ohms) of pathway 2. Use the table to determine the impedance of each parallel circuit. (Hint: You can find the impedance of each pathway by adding the impedance of each component in the pathway.)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Resistor</th>
<th>Inductor</th>
<th>Capacitor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a \Omega$</td>
<td>$-\frac{1}{a}$</td>
<td>$-\frac{1}{b}$</td>
<td>$-\frac{1}{c}$</td>
</tr>
</tbody>
</table>

84. True or False? In Exercises 84–86, determine whether the statement is true or false. Justify your answer.

84. There is no complex number that is equal to its conjugate.
85. $-i\sqrt{6}$ is a solution of $x^4 - x^2 + 14 = 56$.
86. $i^{34} + i^{150} - i^{74} - i^{109} + i^{61} = -1$

87. Express each of the following powers of $i$ as $i$, $-i$, $1$, or $-1$.
   (a) $i^{40}$  (b) $i^{25}$  (c) $i^{50}$  (d) $i^{67}$

88. Prove that the sum of a complex number $a + bi$ and its conjugate is a real number, and that the difference of a complex number $a + bi$ and its conjugate is an imaginary number.
89. Prove that the product of a complex number $a + bi$ and its conjugate is a real number.

Review

In Exercises 90 and 91, write an equation of the line through the point (a) parallel to the given line and (b) perpendicular to the line.

90. $(3, -2)$  $5x - 4y = 8$
91. $(-8, 3)$  $2x + 3y = 5$

In Exercises 92–95, find the x- and y-intercepts of the graph of the equation.

92. $y = -x^2 + 6$
93. $y = x^2 + 2x - 8$
94. $y = |x - 4| + 1$
95. $y = |x| - 1$

96. Mixture Problem  A 5-liter container contains a mixture with a concentration of 50%. How much of this mixture must be withdrawn and replaced by 100% concentrate to bring the mixture up to 60% concentration?
103. (a) \( \frac{1}{4x + x^2} \) (b) \( \frac{4}{x} + \frac{1}{x^2} \)
105. \( f(x) = x^5 - 6x^2 - 19x + 24 \) 107. \( f(x) = x^2 - 4x + 1 \)

**Section 2.4 (page 180)**

1. \( a = -9, b = 4 \) 3. \( a = 6, b = 5 \) 5. \( 4 + 5i \)
7. \( 12, 9, -1 - 5i \) 11. \( -75 \) 13. \( 0.3i \)
15. \( 11 - i \) 17. \( 7 - 3\sqrt{2}i \) 19. \( -14 + 20i \)
21. \( \frac{1}{6} + \frac{7}{6}i \) 23. \( -4.2 + 7.5i \) 25. \( -2\sqrt{3} \)
27. \( -10 \) 29. \( 5 + i \) 31. \( 12 + 30i \)
33. \( 24 \) 35. \( -9 + 40i \)
37. \( \sqrt{-6} - \sqrt{-6} = \sqrt{6}i - \sqrt{6}i = 6i^2 = -6 \) 39. \( 25 \)
41. \( 41 \) 43. \( 484 \) 45. \( 11 \) 47. \( -6i \)
49. \( \frac{16}{41} + \frac{20}{41}i \) 51. \( \frac{3}{2} + \frac{3}{2}i \)
53. \( -7 - 6i \)
55. \( -9 + \frac{10}{168}i + \frac{10}{168}i \) 57. \( -\frac{1}{2} - \frac{3}{2}i \) 59. \( -1 + 6i \)
61. \( -5i \) 63. \( -375\sqrt{3}i \) 65. \( i \) 67. \( 4 + 3i \)
69. \( \frac{1}{\sqrt{3}} \)

**71.**

![Imaginary axis](image)

**73.**

![Imaginary axis](image)

75. Yes. 0, 0, 0, 0, 0, 0
77. Yes.
0.5i, -0.25 + 0.5i, -0.1875 + 0.25i, -0.0273 + 0.4063i,
-0.1643 + 0.4778i, -0.2013 + 0.3430i

**Section 2.5 (page 187)**

1. \( 0, 6, 6 \) 3. \( 2, -4, -4, -4 \) 5. \( -6, i, -i \)
7. \( 2 - 3 + 5i, -3 - 5i \)
9. Zeros: \( 4, -i, i \). One real zero; same
11. Zeros: \( \sqrt{2}i, -\sqrt{2}i, \sqrt{2}i, -\sqrt{2}i \). No real zeros; same
13. \( 2 \pm \sqrt{3} \)
(\( x - 2 - \sqrt{3})(x - 2 + \sqrt{3}) \)
15. \( 6 \pm \sqrt{10} \)
(\( x - 6 - \sqrt{10})(x - 6 + \sqrt{10}) \)
17. \( \pm 5i \)
19. \( \pm 3, \pm 3i \)
(\( x + 5i)(x - 5i) \) (\( x + 3)(x - 3)(x + 3)(x - 3) \)
21. \( 1 \pm i \)
(\( z - 1 + i)(z - 1 - i) \)
23. \( -5, 4 \pm 3i \)
(\( t + 5)(t - 4 + 3i)(t - 4 - 3i) \)
25. \( -\frac{1}{2}, 1 \pm \frac{1}{2}i \)
(\( 4x + 3)(x - 1 - \frac{1}{2}i)(x - 1 + \frac{1}{2}i) \)
27. \( \pm i, \pm 3i \)
(\( x + i)(x - i)(x + 3i)(x - 3i) \)
29. \( 2, 2, \pm 2i \)
(\( x - 2 \)(\( x - 2 \))(\( x - 2 \))(\( x - 2 \)) \)
31. \( -2, -\frac{1}{2}, \pm i \)
(\( x + 2)(x + 1)(x + i)(x + i) \)
33. (a) \( 7 \pm \sqrt{3} \) (b) \( (x^2 - 7)(x - 7 + \sqrt{3}) \)
(c) \( 7 \pm 3\sqrt{3}, 0 \)
35. (a) \( -7 \pm \sqrt{3} \) (b) \( (x^2 + 7)(x + 7 + \sqrt{3}) \)
(c) \( -7 \pm \sqrt{5}, 0 \)
37. (a) \( -6, 3 \pm 4i \) (b) \( (x + 6)(x - 3 - 4i)(x - 3 + 4i) \)
(c) \( -6, 0 \)
39. (a) \( \pm 4i, \pm 3i \) (b) \( (x + 4i)(x - 4i)(x + 3i)(x - 3i) \)
(c) None
41. \( x^3 - x^2 + 25x - 25 \) 43. \( x^3 - 10x^2 + 33x - 34 \)
45. \( x^4 + 37x^2 + 36 \) 47. \( x^4 + 8x^3 + 9x^2 - 10x + 100 \)
49. (a) \( (x^2 + 1)(x^2 - 7) \) (b) \( (x^2 + 1)(x + \sqrt{7})(x - \sqrt{7}) \)
(c) \( x + i(x - i)(x + \sqrt{7})(x - \sqrt{7}) \)
51. (a) \( (x^2 - 6)(x^2 - 2x + 3) \)
(b) \( (x + \sqrt{6})(x - \sqrt{6})(x^2 - 2 + 3) \)
(c) \( (x + \sqrt{6})(x - \sqrt{6})(x - 1 - \sqrt{2})(x - 1 + \sqrt{2}) \)